

Latent Belief Theory and Belief Dependencies: A Solution to the Recovery Problem in the Belief Set Theories

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Abstract

The AGM recovery postulate says: assume a set of propositions X ; assume that it is consistent and that it is closed under logical consequences; remove a belief P from the set minimally, but make sure that the resultant set is again some set of propositions X' which is closed under the logical consequences; now add P again and close the set under the logical consequences; and we should get a set of propositions that contains all the propositions that were in X . This postulate has since met objections; many have observed that it could bear counter-intuitive results. Nevertheless, the attempts that have been made so far to amend it either recovered the postulate in full, had to relinquish the assumption of the logical closure altogether, or else had to introduce fresh controversies of their own. We provide a solution to the recovery paradox in this work. Our theoretical basis is the recently proposed belief theory with latent beliefs (simply the latent belief theory for short). Firstly, through an example, we will illustrate that the vanilla latent belief theory can be made more expressive. We will identify that a latent belief, when it becomes visible, may remain visible only while the beliefs that triggered it into the agent's consciousness are in the agent's belief set. In order that such situations can be also handled, we will enrich the latent belief theory with belief dependencies among attributive beliefs, recording the information as to which belief is supported of its existence by which beliefs. We will show that the enriched latent belief theory does not possess the recovery property. The closure by logical consequences is maintained in the theory, however. Hence it serves as a solution to the open problem in the belief set theories.

1 Introduction

The belief theory with latent beliefs, the latent belief theory for short, was recently proposed [2]. In the framework, every evidence $\{P\}^\diamond$ is a collection of propositions, consisting of one primary proposition P and zero or more attributive propositions expressed in triples: $P(P_1, P_2)$ for some P_1

and P_2 . Each $P(P_1, P_2)$ is basically P_2 in any environment that contains P_1 ; otherwise, it is a latent belief not presently visible, despite its existence, within the environment. What we have called an environment is, in the particular setting of the belief theory, a set of propositions and triples associated to them. Since they characterise the beliefs held by a rational agent, an environment is representative of the state of the mind of a rational agent's, which we may then just call a belief set, as in the AGM belief theory [1]. A logical closure property holds in the latent belief theory: if P_1, \dots, P_n are in a belief set, then any proposition that is a logical consequence of any one or any ones in conjunction of them is also in the belief set. But because the belief sets in the latent belief theory could also contain those triples, they hold more information in general than a belief set in a traditional belief set theory does. To illustrate the point of the triples, suppose that $\{P\}^\diamond$ consists of $P, P(P_1, P_2)$ and $P(P_3, P_4)$. Suppose that an agent believes P_1 , i.e. his/her belief set contains P_1 . Then $P(P_1, P_2)$ is basically P_2 to the agent; and P_2 is in the belief set. But suppose that it does not contain P_3 ; then it is not necessarily the case that P_4 is in the set. Suppose that P_4 is not in the set, then $\{P, P_2\}$ will be the agent's perception of $\{P\}^\diamond$. Nonetheless, if P_3 is added to the set, then the agent's perception of $\{P\}^\diamond$ will be $\{P, P_2, P_4\}$. As this example illustrates, the latent belief theory captures the dynamic nature of a belief/knowledge within the mind of a rational agent's. Some constituents of $\{P\}^\diamond$ are visible, some others may be latent, depending on what beliefs are visible to his/her conscious mind.

Let us contemplate upon the triples. In [2], a latent belief, once it becomes visible to an agent, will acquire the equal significance in footing to any other visible beliefs that he/she holds. In particular, if $P(P_1, P_2)$ is latent to him/her, and if P_2 becomes visible, then contraction of his/her belief set by P does not necessarily entail the loss of P_2 . There are many scenarios that justify the particular behaviour. Consider the following propositions.¹

- | | |
|--|-------------------------------------|
| 1. P_1 : The nerdy-looking boy is Conan. | Kudo, who was a renowned detective. |
| 2. P_2 : There was a high school kid, Shinichi | 3. P_3 : Conan is Shinichi Kudo. |

¹This example is sketched out of Detective Conan.

Suppose the following structure for $\{P_1\}^\diamond$: $\{P_1\}^\diamond = (\{P_1\}, \{P_1(P_2, P_3)\})$, having the primary proposition P_1 as well as one attributive proposition $P_1(P_2, P_3)$. It is not the case that P_1 implies P_2 or P_3 . Neither is it the case P_2 or P_3 P_1 . Now, suppose an agent who, among all the other propositions, believes P_1 , but does not believe either of P_2 and P_3 . That is, suppose that his/her perception of $\{P_1\}^\diamond$ is $\{P_1\}$. When he/she learns P_2 , then P_3 is triggered into his/her mind. His/her perception of $\{P_1\}^\diamond$ is now $\{P_1, P_3\}$. Let us say that his/her belief set is then contracted by P_1 . But there is no reason that P_3 must be also dropped off, even though it was attributive to P_1 when it was latent. He/she does not believe $\{P_1\}^\diamond$ any more; but he/she will still believe P_3 , (or $\{P_3\}^\diamond$ which includes P_3).

1.1 The need for tracking belief dependencies among attributive beliefs

However, there are other cases where the dependency should carry over. Consider the following proposition $\{P_1\}^\diamond$: *Belief changes can be characterised in logic*. Suppose the following propositions.

1. P_1 : Belief changes can be characterised in the AGM belief theory.
2. P_2 : Minimal removal of beliefs is not a random operation.
3. P_3 : Other postulates to the existing AGM postulates characterise belief retention more realistically.

Suppose that $\{P_1\}^\diamond = (\{P_1\}, \{P_1(P_2, P_3)\})$. Suppose some rational agent who knows about the basic AGM postulates but who does not know about the supplementary postulates. Suppose that the agent perceives $\{P_1\}$ for $\{P_1\}^\diamond$. From some psychology magazine, he/she perceives $\{P_2\}^\diamond$. Let us say for simplicity that $\{P_2\}^\diamond = (\{P_2\}, \emptyset)$. Upon his/her accepting it, P_3 , a part of $\{P_1\}^\diamond$ hitherto unknown to him/her, comes into his/her consciousness. But in no time, some external source convinces him/her that the decision on what beliefs remain and what beliefs get removed after a minimal belief contraction is as unpredictable as throwing a die. So he/she drops P_2 . In no time, his/her apprehension grows, and he/she becomes dismissive of logically representing belief changes. The proposition P_1 must go. But so must P_3 , since it has existed on the presumption that it be possible to characterise belief changes in logic, that is, in the AGM theory as he/she perceives it.

1.2 Outlines; on the side theme concerning the recovery property; and other remarks

To bring the extra expressiveness that differentiates the two cases into the latent belief theory, in Section 2 we enrich the theory by defining belief dependencies among attributive beliefs. The basic idea is to extend the definition of an attributive belief to include the fourth parameter: $P(P_1, P_2, n)$ so that n can determine what beliefs P_2 will depend on once it becomes visible. As P_2 is made to exist by P and P_1 , the number of the possibilities is four, and it suffices if n ranges over $\{0, 1, 2, 3\}$. Now, once P_2 becomes visible from

$P(P_1, P_2, n)$, we can say that if $n = 0$, then P_2 is an autonomous belief, independent of P_1 and of P_2 ; if $n = 1$, then it is a dependent belief, independent of P_2 but dependent on P_1 ; if $n = 2$, then it is a dependent belief, independent of P_1 but dependent on P_2 ; and if $n = 3$, then it exists on the existences of both P_1 and P_2 . We store the information of which belief is dependent on what beliefs in a table comprising pairs of a proposition P and a set of propositions Γ on which P is dependent. For instance, it may contain $(P, \{P_1\})$. Suppose that a belief set contains P, P_1 among all the other beliefs, and that it has this table. Then, if the belief set is contracted such that P_1 no longer remains in the resulting belief set, then it must also happen that the set do not contain P which - so does the table say - cannot exist unless P_1 is visible. Whenever a set of new propositions are added to or removed from a belief set, the change is reflected upon the table whose contents are updated appropriately through a set of update postulates.

In Section 3, we present all the belief change postulates, to complete the development of the enriched latent belief theory. We then show that there is no recovery property in our theory. As a brief reminder, the (AGM) recovery postulate says: assume a set of propositions X ; assume that it is consistent and that it is closed under logical consequences; remove a belief P from the set minimally, but make sure that the resultant set is again some set of propositions X' which is closed under the logical consequences; now add P again and close the set under the logical consequences; and we should get a logically closed set of propositions that contains all the propositions that were in X . This postulate has since met objections, many researchers observing that it could bear counter-intuitive results. However, the attempts that have been made so far to amend it either recovered the postulate in full, or else had to introduce fresh controversies of their own [7]. Our theory offers the sought-after solution to the recovery paradox. Section 4 concludes.

2 Formalisation of dependencies among attributive beliefs and of belief sets

Readers may benefit from reading the first few sections of [2]. Although this section is technically self-contained, a detailed intuition is not given to each definition due to space limitation, which is, however, found in the reference. For the intuition of the key new notations as well as examples that illustrate why they are introduced, readers may find it useful to refer back to Section 1.

Let us assume a set of possibly uncountably many atomic propositions. We denote the set by \mathcal{P} , and refer to each element by p with or without a subscript. More general propositions are constructed from \mathcal{P} and the logical connectives of propositional classical logic: $\{\top, \perp, \neg, \wedge, \vee\}$. The subscripts denote the arity. Although the classical implication \supset is not used explicitly, it is derivable from \neg and \vee in the usual manner: $p_1 \supset p_2 \equiv \neg p_1 \vee p_2$. The set of literals, i.e. any p or $\neg p$ for $p \in \mathcal{P}$, is denoted by Lit . The set of all the propositions is denoted by Props ; and each element of Props is referred to by P with or without a subscript. Let us assume that, given any $O \subseteq 2^{\text{Props}}$, $L(O)$ is the set of all the propositions that

are the logical consequences of any (pairs of) elements in O . A set of propositions U is said to be consistent iff for any $P \in \text{Props}$, if $P \in U$, then $\neg P \notin U$; and if $\neg P \in U$, then $P \notin U$. Let us assume that, for any tuples of some sets (U_1, \dots, U_k) , we have $\pi_i((U_1, \dots, U_k)) = U_i$, for $1 \leq i \leq k$. Let us further assume that the union of two tuples of sets: $(U_1, \dots, U_k) \cup (U'_1, \dots, U'_k)$ is $(U_1 \cup U'_1, \dots, U_k \cup U'_k)$.

Definition 1 (Associations and attributive beliefs[2]). *An association tuple is a tuple $(\mathcal{I}, X, \text{Assoc})$. Let \mathfrak{N} be $\{0, 1, 2, 3\}$. Then \mathcal{I} is a mapping from Lit to $2^{\text{Props} \times \text{Props} \times \mathfrak{N}}$. X is an element of 2^{Props} . And Assoc is a mapping from Props to $2^{\text{Props} \times \text{Props} \times \mathfrak{N}}$. Let Exc be a mapping from Props to 2^{Props} such that $\text{Exc}(P) = \mathcal{L}(\{P\}) \cup \{P_1 \in \text{Props} \mid P \in \mathcal{L}(\{P_1\})\}$. Then \mathcal{I} is defined to satisfy that, for any $P \in \text{Lit}$, if either $P_1 \in \text{Exc}(P)$ or $P_2 \in \text{Exc}(P)$, then $(P_1, P_2) \notin \mathcal{I}(P)$. Assoc is defined to satisfy (1) that if P is a tautology, then $\text{Assoc}(P) = (\emptyset, \emptyset, 0)$; (2) that if $\neg P$ is tautology, then $\text{Assoc}(P) = (\text{Props}, \text{Props}, 0)$; and (3) that, if neither;*

- $\text{Assoc}(P) = \mathcal{I}$ if $P \in \text{Lit}$
- $\text{Assoc}(P_1 \wedge P_2) = (\text{Assoc}(P_1) \cup \text{Assoc}(P_2)) \downarrow \text{Exc}(P_1 \wedge P_2)$ where $(U_1, U_2, n) \downarrow U_3 = (U'_1, U'_2, n)$ such that $U'_{1,2} = U_{1,2} \setminus U_3$.
- $\text{Assoc}(P_1 \vee P_2) = \text{Assoc}(P_1 \wedge P_2)$ if $P_1, P_2 \in X$.
- $\text{Assoc}(P_1 \vee P_2) = \text{Assoc}(P_i)$ if $\neg P_j \in X$ for $i, j \in \{1, 2\}, i \neq j$.
- $\text{Assoc}(P_1 \vee P_2) = \text{Assoc}(P_i) \downarrow \text{Exc}(P_1 \wedge P_2)$ if $P_i \in X$ and $P_j, \neg P_j \notin X$ for $i, j \in \{1, 2\}, i \neq j$.
- $\text{Assoc}(P_1 \vee P_2)$ consists of all the pairs (P_x, P_y, n) satisfying the following, otherwise: there exists (P_a, P_A, n_1) in $\text{Assoc}(P_1)$ and there exists (P_b, P_B, n_2) in $\text{Assoc}(P_2)$ such that (1) $P_x = P_a$; (2) $\mathcal{L}(P_a) = \mathcal{L}(P_b)$; (3) $n_1 = n_2$; (4) either $P_B \in \mathcal{L}(P_A)$ or $P_A \in \mathcal{L}(P_B)$; (5) if $P_B \in \mathcal{L}(P_A)$, then $P_y = P_B$, else if $P_A \in \mathcal{L}(P_B)$, then $P_y = P_A$; and (6) $P_x, P_y \notin \text{Exc}(P_1 \wedge P_2)$.
- $\text{Assoc}(\neg(P_1 \wedge P_2)) = \text{Assoc}(\neg P_1 \vee \neg P_2)$.
- $\text{Assoc}(\neg(P_1 \vee P_2)) = \text{Assoc}(\neg P_1 \wedge \neg P_2)$.

We call each $P(P_1, P_2, n)$ for some beliefs P, P_1 and P_2 and some n a belief quadruple, and denote the set of belief quadruples by BQuad . We define the set $\{P(P_1, P_2, n) \in \text{BQuad} \mid [(P_1, P_2, n) \in \text{Assoc}(P)] \wedge^\dagger [(P_1, P_2) \neq (\text{Props}, \text{Props})]\}^2$ to be the set of beliefs attributive to P . We denote the set by $\text{Cond}(P)$. We denote $\bigcup_{P \in \text{Props}'} \text{Cond}(P)$ by $\text{Cond}(\text{Props}')$ where $\text{Props}' \subseteq \text{Props}$. If $\text{Props}' = \text{Props}$, we denote it simply by Cond .

The third condition of the sixth item for disjunction, which says that (P_a, P_A, n_1) in $\text{Assoc}(P_1)$ and (P_b, P_B, n_2) in $\text{Assoc}(P_2)$ are not comparable unless both of the P_A and P_B have the same attributive belief dependency (i.e. $n_1 = n_2$), could be possibly relaxed to be less conservative. We will leave the consideration to a future work.

² In lengthy formal expressions, we use meta-connectives $\wedge^\dagger, \vee^\dagger, \rightarrow^\dagger, \forall, \exists$ in place for conjunction, disjunction, material implication, universal quantification and existential quantification, each following the semantics in classical logic.

2.1 Belief sets and axioms, and update postulates

In our enriched latent belief theory, a belief base is defined to contain a subset of Props and a set of quadruples: $P_a(P_b, P_c, n)$ where $P_a, P_b, P_c \in \text{Props}$ and $n \in \mathfrak{N}$. Additionally, it is defined to contain a table consisting of pairs of $(P, \Gamma) \in \text{Props} \times 2^{\text{Props}}$, which records which belief is dependent on what beliefs. Let us denote a set of the pairs by Π with or without a subscript. Let us call some tuple (Γ, Δ, Π) for some $\Gamma \in 2^{\text{Props}} \setminus \emptyset$, some $\Delta \in 2^{\text{BQuad}}$ and some Π a belief base. We denote the set of belief bases by BBase , and refer to each element by B with or without a subscript. A belief set is defined to be an element of BBase that satisfies the following axioms.

1. $\mathcal{L}(\pi_1(B)) = \pi_1(B)$ (Logical closure).
2. If $P \in \pi_1(B)$, then there is a finite subset X of $\pi_1(B)$ such that $P \in \mathcal{L}(X)$. (Compactness).
3. If $P \notin \pi_1(B)$, then for any $P_1, P_2 \in \text{Props}$ and any $n \in \mathfrak{N}$ it holds that $P(P_1, P_2, n) \notin \pi_2(B)$ (Attributive belief adequacy).
4. If $P \in \pi_1(B)$, then $(P, \Gamma) \in \pi_3(B)$ for some Γ (Support adequacy 1).
5. If $P \notin \pi_1(B)$, then $(P, \Gamma) \notin \pi_3(B)$ for any Γ (Support adequacy 2).
6. If $(P, \Gamma) \in \pi_3(B)$, then $\Gamma \neq \emptyset$ (Support sanity).
7. If $(P_1, \Gamma_1), (P_2, \Gamma_2), (P_1 \vee P_2, \Gamma) \in \pi_3(B)$, then $\mathcal{L}(\Gamma_1) \cup \mathcal{L}(\Gamma_2) = \mathcal{L}(\Gamma)$ (Disjunctive support propagation).
8. If $(P_1, \Gamma_1), (P_2, \Gamma_2), (P_1 \wedge P_2, \Gamma) \in \pi_3(B)$, then $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2) = \mathcal{L}(\Gamma)$ (Conjunctive support propagation).
9. If $(P_1, \Gamma_1), (P_2, \Gamma_2) \in \pi_3(B)$ such that $\mathcal{L}(P_1) \subseteq \mathcal{L}(P_2)$, then $\mathcal{L}(\Gamma_2) \subseteq \mathcal{L}(\Gamma_1)$ (Support monotonicity).
10. If P is a tautology such that $P \in \pi_1(B)$, then if $(P, \Gamma) \in \pi_3(B)$, then $\top \in \Gamma$ (Tautological support).

Compared to the definition of a belief set as found in [2], this definition does not conduct the fixpoint iterations. For the axioms around the third component of a belief base (the items from 4 to 10), insertion of a couple of notes here may be useful. The (Support adequacy 1), the (Support adequacy 2) and the (Support sanity) ensure that if B is a belief set, then that P is in $\pi_1(B)$ means that (P, Γ) for a non-empty $\Gamma \in 2^{\text{Props}}$ is in $\pi_3(B)$, and vice versa. The (Disjunctive support propagation) and the (Conjunctive support propagation) say how supporting propositions are determined, deterministically up to \mathcal{L} , along the \mathcal{L} ladders. The basic functionality of the (Support monotonicity) is to make $\mathcal{L}(\Gamma_1) = \mathcal{L}(\Gamma_2)$ in case P_1 and P_2 are indistinguishable in \mathcal{L} . Finally the (Tautological support) effectively states that a tautological belief in a belief set is independent of any non-tautological beliefs. We say that a belief base B , a belief set in particular, is consistent iff $\pi_1(B)$ is consistent.

The following sets of postulates: one for when an item is added to it, as characterised by the operator \circ_B for $B \in \text{BBase}$ which takes a belief base and a pair of a proposition and a set of propositions to return a belief base; and one for when an item is removed from it, as characterised by the operator $-_B$ for $B \in \text{BBase}$ which takes a belief base and a set

of propositions to return a belief base, define rules for updating the support table (which is the third component of a belief base).

Support table augmentation operator \circ satisfies the following.

1. For each $(P, \Gamma) \in \Pi_1$, if $(P, \Gamma_x) \in \pi_3(B)$, then $(P, \Gamma_x \cup \Gamma \cup \bigcup_{\{(P_1, \Gamma_1) \in \Pi_1 \mid L(P) \subseteq L(P_1)\}} \Gamma_1) \in \pi_3(B \circ_B \Pi_1)$.
2. For each $(P, \Gamma) \in \Pi_1$, if $(P, \Gamma_x) \notin \pi_3(B)$ for no Γ_x , then $(P, \Gamma \cup \bigcup_{\{(P_1, \Gamma_1) \in \Pi_1 \mid L(P) \subseteq L(P_1)\}} \Gamma_1) \in \pi_3(B \circ_B \Pi_1)$.
3. If $(P_x, \Gamma_x) \in \pi_3(B)$ and if, for all $(P, \Gamma) \in \Pi_1$, it holds that $P_x \notin \text{Exc}(P)$, then $(P_x, \Gamma_x) \in \pi_3(B \circ_B \Pi_1)$.
4. If $(P, \Gamma) \notin \pi_3(B)$, and if $(P_1, \Gamma_1) \notin \Pi_1$ for any Γ_1 and for any P_1 such that $L(\{P_1\}) = L(\{P\}) \neq L(\{\top\})$, then $(P_2, \Gamma_2) \notin \pi_3(B \circ_B \Pi_1)$ for any Γ_2 and any P_2 such that $L(P_2) = L(P)$.
5. $\pi_i(B) = \pi_i(B \circ_B \Pi_1)$ for $i \in \{1, 2\}$.
6. $B \circ_B (P, \Gamma)$ satisfies the earlier axioms.

To explain these a little, the fifth postulate ensures that the operation \circ_B acts, if any, only upon $\pi_3(B)$, leaving $\pi_{1,2}(B)$ intact. Because of this, whatever changes that \circ_B operation would make to $\pi_3(B)$, the sixth postulate guarantees, through the (Support adequacy 1) and the (Support adequacy 2), that any $(P, \Gamma) \in \pi_3(B \circ_B \Pi_1)$ is linked to $P \in \pi_1(B \circ_B \Pi_1)$: it is in $\pi_3(B \circ_B \Pi_1)$ only if $P \in \pi_1(B \circ_B \Pi_1)$. Now, as to what those supporting sets of beliefs are for each belief, if B satisfies the axioms 7 - 10, then determining the supporting set of beliefs for each of key beliefs suffices to determine all the other supporting sets for each belief, due again to the sixth postulate which includes (Disjunctive support propagation) and (Conjunctive support propagation). As for what the key beliefs are, there are those beliefs in B that are unrelated to the elements of Π_1 by L . Their supporting set of beliefs should not change, which is ensured in the third postulate. Apart from those beliefs, it suffices to ensure the first two postulates to determine all the other supporting sets. Finally, the fourth postulate ensures that the change to B should be minimal by the \circ_B operation.

Support table reduction operator $-$ satisfies the following. Here $X \setminus Y$ denotes $\{P \in X \mid [L(\{P\}) = L(\{\top\})] \vee^\dagger [\neg \exists P_1 \in Y. L(\{P_1\}) = L(\{P\})]\}$.

1. $[(P_1, \Gamma_1 \setminus \Gamma) \in \pi_3(B -_B \Gamma)]$ if $[(P_1, \Gamma_1) \in \pi_3(B)]$.
2. If $(P_1, \Gamma_1) \in \pi_3(B -_B \Gamma)$, then $(P_1, \Gamma_x) \in \pi_3(B)$ for $\Gamma_x = \Gamma_1 \cup \Gamma$ or else $\Gamma_x = \Gamma_1$.
3. $\pi_i(B) = \pi_i(B -_B \Gamma)$ for $i \in \{1, 2\}$.
4. $B -_B \Gamma$ satisfies the earlier axioms except for (Support sanity).

The reduction operation is simpler, trying to remove matching elements off each $(P, \Gamma) \in \pi_3(B)$. Then the earlier axioms, in particular the (Support adequacy 1) and the (Support adequacy 2) link the elements to the first component of B . The reason that we do not include the (Support sanity) in the fourth postulate is just so that we can get the belief contraction operation right in the next section.

3 Belief change postulates

Let us define that \boxplus is the belief expansion operator in our belief theory, \equiv the belief contraction operator, and \star the belief revision operator. We require each one of them to be a fixpoint iterating process. Since the belief revision operation has been always a derivable operation in the AGM tradition, we only define \boxplus and \equiv ; and, later on, will show how the two operations are combined into \star . The graphical representations of $B_{\text{init}} \boxplus \{P\}^\diamond$ and $B_{\text{init}} \equiv \{P\}^\diamond$, assuming that B_{init} is a belief set, are found in Figure 1.³ The $\text{Visible}_B(\{P\}^\diamond)$ comes from [2]. It is $P \cup \{P_2 \mid [P(P_1, P_2, n) \in \pi_2(\{P\}^\diamond)] \wedge^\dagger [P_1 \in \pi_1(B)]\}$. In both of the diagrams, $X := Y$ denotes the assignment of Y to X . $\Gamma := X; B_0 := Y$ means that the assignment operations are taken in sequence: the assignment to Γ , followed by that to B_0 . Both of the processes continue until the fixpoint is reached.

We now define all the participants in the two representations. Before moving further, we recall ([2]) the postulate for the association tuple.

Association tuple has one postulate:

1. Each B has an association tuple $(\mathcal{I}, \pi_1(B), \text{Assoc})$ for some \mathcal{I} and some **Assoc**.

Internal expansion operator $+$ has two postulates:

1. $B + \Gamma = (L(\pi_1(B) \cup \Gamma), \text{Cond}(\pi_1(B + \Gamma)), \pi_3(B))$.
2. If the association tuple for B is $(\mathcal{I}, \pi_1(B), \text{Assoc})$, then that for $B + \Gamma$ is $(\mathcal{I}, \pi_1(B + \Gamma), \text{Assoc})$.

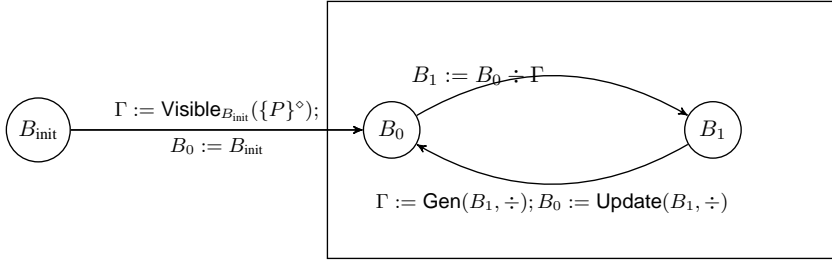
Internal contraction operator \div has the following postulates:

1. $B \div \Gamma = (L(B \div \Gamma), \text{Cond}(\pi_1(B \div \Gamma)), \pi_3(B))$.
2. $\forall P_1 \in \Gamma. P_1 \notin L(\top) \rightarrow^\dagger \pi_1(B \div \Gamma)$.
3. $\pi_1(B \div \Gamma) \subseteq \pi_1(B)$.
4. $(\forall P_1 \in \Gamma. P_1 \notin \pi_1(B) \vee^\dagger P_1 \in L(\top)) \rightarrow^\dagger B \div \Gamma = B$.
5. $[L(\Gamma_1) = L(\Gamma_2)] \rightarrow^\dagger [B \div \Gamma_1 = B \div \Gamma_2]$.
6. $B \subseteq (B \div \Gamma) + \Gamma$.
7. If the association tuple for B is $(\mathcal{I}, \pi_1(B), \text{Assoc})$, then that for $B \div \Gamma$ is $(\mathcal{I}, \pi_1(B \div \Gamma), \text{Assoc})$ (Association update).

These postulates closely coincide with the AGM postulates [1]. In passing, two more postulates may be added to the list above: $\forall P_1 \wedge P_2 \in \Gamma. [P_1 \notin Cn(B) \div \Gamma] \rightarrow^\dagger [Cn(B) \div \Gamma \subseteq Cn(B) \div \Gamma(P_1 \wedge P_2 \mapsto P_1)]$; and $\forall P_1 \wedge P_2 \in \Gamma. (Cn(B) \div \Gamma(P_1 \wedge P_2 \mapsto P_1)) \cap (Cn(B) \div \Gamma(P_1 \wedge P_2 \mapsto P_2)) \subseteq Cn(B) \div \Gamma. \Gamma(P_1 \wedge P_2 \mapsto P_x)$ means to replace all the occurrences of $P_1 \wedge P_2 \in \Gamma$ with P_x . The two postulates are intended to regulate belief retention [1]. We are hardly concerned with these supplementary postulates in this particular work, but a mentioning of them may be useful to a reader who is interested in retention of the beliefs based on the concept of the epistemic entrenchment. Somehow related to it, recall ([1]) that, generally speaking neither $B \div \Gamma$ nor $B + \Gamma$ is a deterministic

³ Although the inner components have not yet been formally defined, we believe that the display of the visual representations, and then detailing the concepts used there finely will better explain these operations.

$$B_{\text{init}} \doteq \{P\}^\diamond \quad (\text{Belief contraction})$$



$$B_{\text{init}} \boxtimes \{P\}^\diamond \quad (\text{Belief expansion})$$

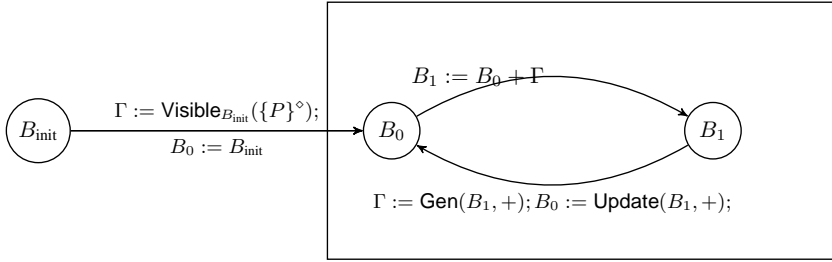


Figure 1: Graphical representations of the belief expansion $B_{\text{init}} \boxtimes \{P\}^\diamond$ and the belief contraction $B_{\text{init}} \doteq \{P\}^\diamond$ in our latent belief theory. B_{init} is assumed to be a consistent belief set, although the restriction is more a pragmatic than a technical one.

operation.

Gen and Update are defined as follows.

$\text{Gen}(B, \div) := \{P \mid [(P, \Gamma) \in \pi_3(B)] \wedge^\dagger (\forall P_1 \in \Gamma. P_1 \notin \pi_1(B)) \vee^\dagger [\Gamma = \emptyset]\}$. *Explanation:* \div may remove beliefs off the first component of a belief base. Suppose some beliefs are indeed dropped off, and that we have $B_1 \subset B_0$ (see the graphical representation). Now, it could be that some proposition $P_x \in B_1$ may have lost all the propositions for supporting its existence. Then P_x can no longer subsist in B_1 , which will be further contracted by all such P_x in the next round of the fixpoint iteration.

$\text{Gen}(B, +) := \{P_2 \mid [P(P_1, P_2, n) \in \pi_2(B)] \wedge^\dagger [P, P_1 \in \pi_1(B)]\}$. *Explanation:* When a belief base is augmented with a new set of beliefs, it could happen that latent beliefs become visible, which is a subset of all the propositions generated by this set construction.

$\text{Update}(B, \div) := B -_B \{P \mid [(P, \Gamma) \in \pi_3(B)] \wedge^\dagger [P \notin \pi_1(B)]\}$. *Explanation:* The support table is updated to reflect the loss of beliefs by \div . Specifically, any element in $\pi_3(B)$ is removed if the first component of the element is no longer in $\pi_1(B)$. However, recall that the $-$ operation does not satisfy the (Support sanity) axiom. Hence even if the operation should generate some (P, Γ) such that $\Gamma = \emptyset$, it is not removed from the third component.

$\text{Update}(B, +) := B \circ_B \{(P_2, \Gamma) \mid [P(P_1, P_2, n) \in \pi_2(B)] \wedge^\dagger [\Gamma = \rho(P(P_1, P_2, n))]\}$ where $\rho(P(P_1, P_2, n))$ is $\{\top\}$ if $n = 0$; is $\{P\}$ if $n = 1$; is $\{P_1\}$ if $n = 2$; and is $\{P, P_1\}$ if $n = 3$. There is certain difficulty in having an intuitively appealing representation theorem of \doteq , non-iteratively. In the AGM be-

lief contraction operation, the belief set as a set of propositions, say X , is contracted by some proposition P . No matter how many candidates are for the result of the contraction operation, the candidates are determined by X and P alone with no other non-deterministic factors. However, in our characterisation, the $(k + 1)$ -th fixpoint iteration depends upon the result of the k -th internal contraction by \div , which can be known only non-deterministically. This makes it hard to generate a set-based representation of the contraction operation such that it retain the same intuitive appeal as the AGM representation theorem does. For this reason, we regard the transition systems shown earlier as the representation theorem equivalents for \doteq and \boxtimes . By contrast, the internal operations by $+$ and \div almost exactly emulate the AGM expansion and contraction operators (Cf. [1]), and the set-based representation is feasible for each of them without costing intuitive appeal. Particularly for \div , it goes as follows [1]. Suppose that B is a belief set. Then, $\pi_1(B) \div \Gamma = \bigcap (\gamma(\Xi(\pi_1(B), \Gamma)))$. Ξ is a mapping from belief sets and propositions into belief sets. For any belief set B and any Γ , we say that a belief set B_1 satisfying $\pi_1(B_1) \subseteq \pi_1(B)$ is a maximal subset of $\pi_1(B)$ for Γ iff

1. For any $P_1 \in \Gamma$, $P_1 \notin \pi_1(B_1)$ if P_1 is not a tautology.
2. For any belief set B_2 , if $\pi_1(B_1) \subset \pi_1(B_2) \subseteq \pi_1(B)$, then there exists some $P_a \in \Gamma$ such that $P_a \in \pi_1(B_2)$.

We define $\Xi(\pi_1(B), \Gamma)$ to be the set of all the subsets of $\pi_1(B)$ maximal for Γ . We further define a function γ , so that, if $\Xi(\pi_1(B), \Gamma)$ is not empty, then $\gamma(\Xi(\pi_1(B), \Gamma))$ is a subset of $\Xi(\pi_1(B), \Gamma)$; or if it is empty, it is simply $\pi_1(B)$.

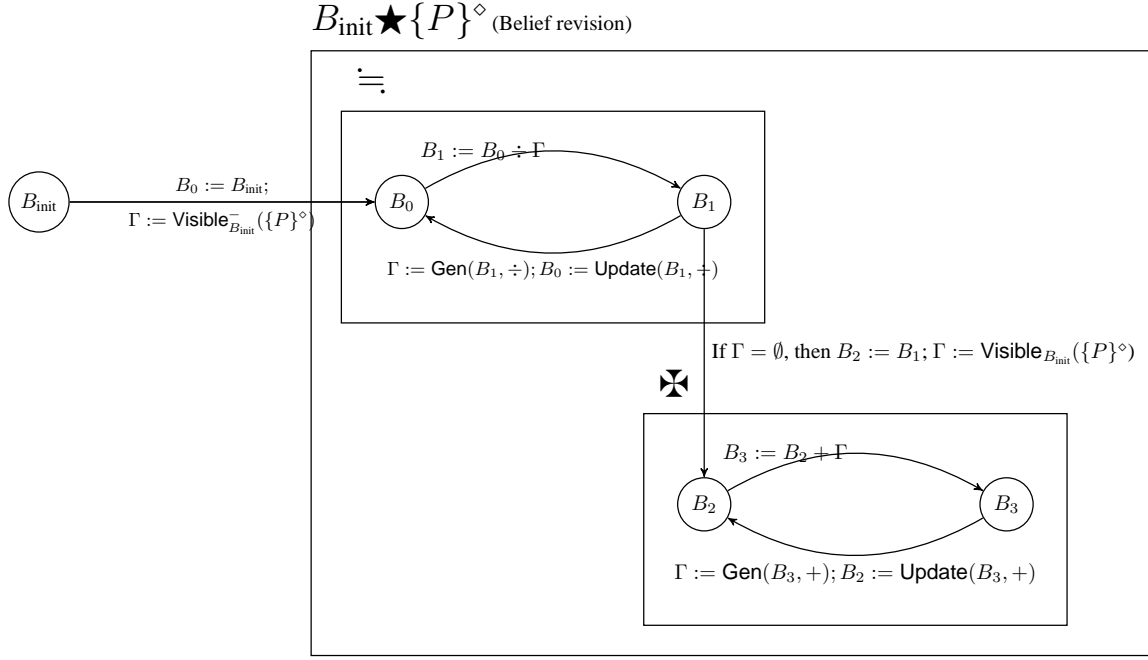


Figure 2: The belief revision operation $B_{\text{init}} \star \{P\}^\diamond$ as a composition of the belief contraction and expansion.

Then we have that $\pi_1(B) \div \Gamma = \bigcap (\gamma(\Xi(\pi_1(B), \Gamma)))$.

From \bowtie and \equiv , we define the belief revision operator:

$$B \star \{P\}^\diamond = (B \equiv (\text{Visible}_B^{-}(\{P\}^\diamond), \emptyset) \bowtie (\text{Visible}_B(\{P\}^\diamond), \text{Cond}(\text{Visible}_B(\{P\}^\diamond))).$$

And we have the representation in the transition system, as shown in Figure 2. For the same reason that has hindered us from having a set-based representation of \equiv , without losing an appeal to intuition, it is difficult to come up with non-iterative postulates for \star . This is not the sign that our theory is not robust. It just confirms the point that every belief change operation is an iterative process in our theory.

Theorem 1 (Preservation). *Let B be a consistent belief set. Then $B \bowtie \{P\}^\diamond$, $B \equiv \{P\}^\diamond$ and $B \star \{P\}^\diamond$ are again a belief set.*

Let us also note the following important result.

Theorem 2 (No recovery). *Let B be some belief set, and let $\{P\}^\diamond$ be some external information. Then we can find a pair of $B, \{P\}^\diamond$ such that both of the following fail to hold. $\pi_1(B) \subseteq \pi_1((B \equiv \{P\}^\diamond) \bowtie \{P\}^\diamond)$. $\pi_1(B) \subseteq \pi_1((B \equiv \{P\}^\diamond) \bowtie (\text{Visible}_B(\{P\}^\diamond), \emptyset))$.*

Proof. The first weaker result holds already in the vanilla latent belief theory; Cf. [2]. To give the evidence that $\pi_1(B) \not\subseteq \pi_1((B \equiv \{P\}^\diamond) \bowtie (\text{Visible}_B(\{P\}^\diamond, \emptyset)))$, suppose that our language is constructed from p_1, p_2, p_3 and the logical connectives $\{\top, \perp, \wedge, \vee\}$. Suppose that no pairs of the three propositions are associated by the logical consequences. Now, suppose that B is a belief set; $\pi_1(B) = \mathcal{L}(\{p_1, p_3\})$; $\pi_2(B)$ contains $p_1(p_2, p_3, 1)$ but does not contain any $P_x(P_y, P_z, n)$ such that $\mathcal{L}(P_z) \subseteq \mathcal{L}(p_3)$ or that $\mathcal{L}(p_3) \subseteq \mathcal{L}(P_z)$; and that $\pi_3(B)$ contains $(p_3, \{p_1\})$, among others. Then $p_3 \notin (B \equiv \{p_1\}^\diamond)$. Suppose that $B \equiv \{p_1\}^\diamond$

does not retain any logical consequence of p_3 apart from tautological propositions. Then the belief set contains no propositions P_α such that $\mathcal{L}(P_\alpha) = \mathcal{L}(P_z \supset p_3) \neq \mathcal{L}(\top)$. Then $p_3 \notin (B \equiv \{p_1\}^\diamond) \bowtie (\{p_1\}, \emptyset)$, as required. \square

This result can be also adapted to the AGM belief theory, even though there are no quadruples in the AGM setting, so long as the same dependencies among propositions are facilitated in the AGM operations of belief expansion and contraction. With this remark, we are positive that this work has truly offered a satisfactory solution to the recovery paradox as far as the cases similar to the scenarios in the opening examples are concerned.

4 Conclusion

We have presented an enriched latent belief theory, in which belief dependencies based on how a latent belief has become visible to agents' belief sets can be expressed. We have shown that there is no recovery postulate in our belief theory, thus giving a solution to the recovery paradox that has been in the belief set theories. Our theory indicates that the time may have come to study beyond the Gärdenfors' principle [3]: "If a belief state is revised by a sentence A , then all sentences in K that are independent of the validity of A should be retained in the revised state of belief", upon which many current works on dependencies, such as [6] for a recent one, appear to be based. The examples in Section 1 in any case indicate that beliefs, even though not connected by logical consequences, can still be related by their contents, which is the standpoint that has been already taken in [2]. Although we had to present just the result, there appears to be some observation that has not been detailed in the literature around the recovery paradox. In another work of ours, which is going to be much

less technical, we will have an overview of the paradox: what it was, and how it was naturally resolved. One future work will focus on applications of this enriched latent belief theory. Connection to theories of concurrency in computer science will be sought after.

Related works

Several works questioned the AGM recovery postulate [4; 5; 7]. The attempt to amend it has not been successful within the belief set setting, however. There are works on belief revision with non-classical logics that do not have the classical logical consequence relations. Not surprisingly, the recovery property does not necessarily hold in the setting, as evidenced also in the belief base theory, which was cultivated notably by Hansson and others.

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